UPR-1235-T
DAMTP-2012-2

Conformal Symmetry of a Black Hole as a Scaling Limit: A Black Hole in an Asymptotically Conical Box

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May 10, 2012

Abstract

We show that the previously obtained subtracted geometry of four-dimensional asymptotically flat multi-charged rotating black holes, whose massless wave equation exhibit $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry may be obtained by a suitable scaling limit of certain asymptotically flat multi-charged rotating black holes, which is reminiscent of near-extreme black holes in the dilute gas approximation. The co-homogeneity-two geometry is supported by a dilation field and two gauge-field strengths. We also point out that these subtracted geometries can be obtained as a particular Harrison transformation of the original black holes. Furthermore the subtracted metrics are asymptotically conical (AC), like global monopoles, thus describing "a black hole in an AC box". Finally we account for the the emergence of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry as a consequence of the subtracted metrics being Kaluza-Klein type quotients of $AdS_3 \times 4S^3$. We demonstrate that similar properties hold for five-dimensional black holes.

Contents

1 Introduction

The microscopic entropy of (near)-supersymmetric asymptotically flat black holes has been well understood in terms of weakly coupled two-dimensional conformal field theory (See, e.g., the review [1] and references therein). On the other hand, general multi-charged rotating black holes in four [2] and five [3] dimensions have an entropy formula [2] strongly suggestive of a possible microscopic interpretation in terms of a weakly coupled two-dimensional conformal field theory. Some early work along these directions was pursued in [4, 5, 6]. [There are indications that general asymptotically anti-deSitter black holes may also have a microscopic description, as indicated by the quantized value of the product of horizon areas [7].]

An intriguing clue to the internal structure of a black hole is the structure of the wave equation in its background. The wave equation for a massless scalar field turns out to have remarkable simplifications even for general multi-charged rotating black holes [5, 6] . In particular the equation is separable and it has an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry, when certain terms are subtracted. It turns out that these terms can be neglected in many special cases, including the near-supersymmetric limit (the AdS/CFT correspondence) [8, 5, 6], the near extreme rotating limit (the Kerr/CFT correspondence) [9, 10], and the low energy limit [5, 11]. However, in general there is no $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ isometry of the subtracted metrics. In [11] it is asserted that the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry suggested by the massless wave equation is there–it is just that it is spontaneously broken ("hidden conformal symmetry").

Recently, in [12] an explicit part of the general multi-charged rotating black hole geometry, which exhibits the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry of the the wave equation, was constructed. Since this metric differs from the original black hole metric by removing certain terms in the warp factor, only, it was dubbed the "subtracted geometry". The subtracted metric has the same horizon area and periodicity's of the angular and time coordinates in the near horizon regions. It is thus expected to preserve the internal structure of the black hole. The subtracted geometry is not asymptotically flat (AF) but is asymptotically conical, AC, and admits a Lifshitz type homothety which scales space and time differently. The physical interpretation of this subtraction is the removal of the ambient asymptotically Minkowski space-time in a way that extracts the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry of the black hole ("black hole in an AC confining box").

The subtraction has been explicitly implemented both for the five-dimensional three-charge rotating black holes [12] and four-dimensional four-charge ones [13]. For four-dimensional black holes the subtracted geometry is a Kaluza-Klein type quotient of $AdS_3 \times 4S^2$ with $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry manifest. [Analogously, for five-dimensional black holes the subtracted geometry is as Kaluza-Klein type quotient of $AdS_3 \times S^3$ with $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(4)$ symmetry manifest.] In [13] the explicit sources for static subtracted geometry were obtained as a part of the so-called STU-model (sector of the four-dimensional N=2 supergravity coupled to three vector supermultiplets) and its lift to five dimensions corresponds to the minimal supergravity.

In this paper we further elucidate the origin and the geometric interpretation of the subtracted geometries. In particular we realize these geometries for general four-charge rotating solutions as a scaling limit of certain multi-charge rotating black holes with three large charges, reminiscent of the near-extreme multi-charge rotating black holes in the dilute gas approximation. This procedure fully determines the sources in the rotating multi-charge cases both in four and five dimensions.

We also demonstrate that the subtracted geometry of the Schwarzschild solution can be obtained by performing a Harrison transformation on the original Schwarzschild solution. This procedure confirms tat the subtracted geometry is a solution of the STU-model. In the previous

version of this paper we conjectured that the subtracted geometry of general multi-charged rotating black holes arises as a Harrison transformation of the original multi-charged rotating black hole. This has since been confirmed [14]. These results show that the original black hole and the subtracted geometry clearly lie in the same duality orbit, specified by a Harrison transformation and passing through the original black hole. Thus any physical property of the original black hole solution which is invariant under duality transformation of the theory remains a property of the subtracted geometry.

Furthermore we analyze the asymptotic structure of the subtracted metrics and find, because the energy density falls of inversely as the second power of the radial distance that they are asymptotically conical (AC), rather than asymptotically flat, in a way which is similar to the asymptotic behaviour of global monopoles and isothermal gas spheres. Since the metric component g_{tt} is proportional to the 6th power of the radial area distance, the subtracted metrics exhibit confining behaviour analogous to that found in AdS_4, justifying their interpretation as "a black hole in an AC confining box". The spatial dependence of the dilaton implies that the gauge coupling constants run logarithmically in the radial direction, not even stabilizing at infinity. This behaviour, together with energy densities falling off inversely as the square of the radial distance is shown to persist in Dilaton-Maxwell theory when a limit of vanishing Newton's constant is taken.

The paper is organized in the following way: In Section 2 we obtain the subtracted geometry of general four-charge rotating black holes (of $\mathcal{N} = 2$ supergravity coupled to three vector supermultiplets) as a scaling limit of black holes holes with three large charges, accounting for all the sources both in the static (Section 2.1) and rotating (Section 2.2) case. In Section 2.3 we demonstrate for the Schwarzschild solution that the subtracted geometry emerges as a c Harrison transformation on the unsubtracted one. In Section 3 we further discuss asymptotics of the subtracted geometry and draw a comparison with other AC examples in General Relativity, in particular those of isothermal gas spheres and a global monopole (Section 3.1). The confining properties of the AC metrics are discussed in Section 3.2. In Section 4 we discuss the local and global properties of the geometry which is a Kaluze-Klein coset of $AdS_2 \times 4S^2$ (Section 4.1), derive the form of the isometry generators for the geometry lifted to five-dimensions (Section 4.2) and discuss isometries of the five-dimensional metric constrained to special slices (Section 4.3). Conclusions are given in Section 5. In Appendix we present the full solution of the subtracted geometry for general three-charge black holes in five dimensions.

2 Subtracted Geometry as a Scaling Limit

In this section we obtain the explicit expressions for sources that support the subtracted geometry of general four-charge rotating black holes by taking a scaling limit of a rotating black hole with three equal large charges and the fourth finite one. The limit is closely related to the "dilute gas" (near-BPS) rotating black hole solution.

The original four-charge rotating solution [2], along with the explicit expressions for all four gauge potentials was given in [15] as a solution of the bosonic sector four-dimensional Lagrangian density of the $\mathcal{N} = 2$ supergravity coupled to three vector supermultiplets [1]:

$$\mathcal{L}_4 \;=\; R\,{*\mathbf{1}} - \frac{1}{2}{*d\varphi_i} \wedge d\varphi_i - \frac{1}{2}e^{2\varphi_i}\,{*d\chi_i} \wedge d\chi_i - \frac{1}{2}e^{-\varphi_1}\,(e^{\varphi_2-\varphi_3}\,{*F_1} \wedge F_1$$

[1]Conventions for dualisation in [15] are that a p-form ω with components defined by $\omega = 1/p!\,\omega_{i_1\cdots i_p}\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ has dual $*\omega$ with components $(*\omega)_{i_i\cdots i_{D-p}} = 1/p!\,\epsilon_{i_1\cdots i_{D-p}j_i\cdots j_p}\,\omega^{j_1\cdots j_p}$. Note also that the normalization of the gauge field strengths above and in [15] differs from the standard one by a factor of $\sqrt{2}$.

$$
\begin{aligned}
&+ \quad e^{\varphi_2+\varphi_3} *F_2 \wedge F_2 + e^{-\varphi_2+\varphi_3} *\mathcal{F}_1 \wedge \mathcal{F}_1 + e^{-\varphi_2-\varphi_3} *\mathcal{F}_2 \wedge \mathcal{F}_2) \\
&- \quad \chi_1 \left(F_1 \wedge \mathcal{F}_1 + F_2 \wedge \mathcal{F}_2\right) , \qquad (1)
\end{aligned}
$$

where the index i labelling the dilatons φ_i and axions χ_i ranges over $1 \leq i \leq 3$. The four U(1) field strengths can be written in terms of potentials as

$$
\begin{aligned}
F_1 &= dA_1 - \chi_2\, d\mathcal{A}_2 \, , \\
F_2 &= dA_2 + \chi_2\, d\mathcal{A}_1 - \chi_3\, dA_1 + \chi_2\, \chi_3\, d\mathcal{A}_2 \, , \\
\mathcal{F}_1 &= d\mathcal{A}_1 + \chi_3\, d\mathcal{A}_2 \, , \\
\mathcal{F}_2 &= d\mathcal{A}_2 \, .
\end{aligned}
$$

The four-dimensional theory can be obtained from six-dimensions, by reducing on a two-torus the following action (See, e.g., [15] and references therein.):

$$
\mathcal{L}_6 = R *\mathbf{1} - \frac{1}{2} *d\phi \wedge d\phi - \frac{1}{2} e^{-\sqrt{2}\phi} *F_{(3)} \wedge F_{(3)} \qquad (2)
$$

The above six-dimensional action is a consistent truncation of the Neveu-Schwarz Neveu-Schwarz sector of toroidally compactified superstring theory.

In the following we shall employ the form of the gauge potentials A_I $(I = 1, 2, 3, 4)$ which define: $*F_1$, F_2, $*\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. (In the static case these gauge potentials all correspond to electric fields.) The four-charge rotating solution [2] with all the sources explicitly displayed was given in [15] [2] . Here we display the metric, only:

$$
ds_4^2 = -\Delta_0^{-1/2} G (dt + \mathcal{A})^2 + \Delta_0^{1/2} \left(\tfrac{dr^2}{X} + d\theta^2 + \tfrac{X}{G} \sin^2\theta d\phi^2 \right) \, , \qquad (3)
$$

where

$$
\begin{aligned}
X &= r^2 - 2mr + a^2 \, , \\
G &= r^2 - 2mr + a^2 \cos^2\theta \, , \\
\mathcal{A} &= \frac{2ma \sin^2\theta}{G} \left[(\Pi_c - \Pi_s) r + 2m \Pi_s \right] d\phi \, , \qquad (4)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_0 = \quad & \prod_{I=1}^{4} (r + 2m \sinh^2\delta_I) + 2a^2 \cos^2\theta [r^2 + mr \sum_{I=1}^{4} \sinh^2\delta_I + \; 4m^2 (\Pi_c - \Pi_s) \Pi_s \\
& -2m^2 \sum_{I<J<K} \sinh^2\delta_I \sinh^2\delta_J \sinh^2\delta_K] + a^4 \cos^4\theta \, . \qquad (5)
\end{aligned}
$$

We are employing the following abbreviations:

$$
\Pi_c \equiv \prod_{I=1}^{4} \cosh\delta_I \, , \quad \Pi_s \equiv \prod_{I=1}^{4} \sinh\delta_I \, . \qquad (6)
$$

The solution is parameterised by the bare mass parameter m, the rotational parameter a and four charge parameters δ_I $(I = 1, 2, 3, 4)$. The solution is written as a U(1) fibration over the

[2]Black hole solutions of the Lagrangian density (1) are generating solutions of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity theory, which can be obtained as a toroidal compactification on an effective heterotic string theory and Type IIA superstring theory, respectively. The full set of solutions can be obtained by acting with a subset of respective $\{S, T\}$- and U- duality transformations. (See e.g., [16].)

three dimensional base, independent of the charge parameters, and the warp factor denoted by Δ_0.

In the static case one sets $a = 0$ and the solution simplifies significantly [17]:

$$ds_4^2 \quad = -\Delta_{0s}^{-1/2} X dt^2 + \Delta_{0s}^{1/2} \left(\tfrac{dr^2}{X} + d\theta^2 + \sin^2\theta d\phi^2 \right) \ , \tag{7}$$

where

$$X \quad = \quad r^2 - 2mr \ , \tag{8}$$

$$\Delta_{0s} \quad = \quad \prod_{I=1}^{4} (r + 2m \sinh^2 \delta_I) \ , \tag{9}$$

and the scalar fields and the gauge potentials take the form:

$$\begin{aligned}
&\chi_i = 0 \, , \quad e^{\varphi_1} = \left[\frac{(r + 2m\sinh^2\delta_1)(r + 2m\sinh^2\delta_3)}{(r + 2m\sinh^2\delta_2)(r + 2m\sinh^2\delta_4)} \right]^{\frac{1}{2}} \ , \\
&e^{\varphi_2} = \left[\frac{(r + 2m\sinh^2\delta_2)(r + 2m\sinh^2\delta_3)}{(r + 2m\sinh^2\delta_1)(r + 2m\sinh^2\delta_4)} \right]^{\frac{1}{2}} \ , \quad e^{\varphi_3} = \left[\frac{(r + 2m\sinh^2\delta_1)(r + 2m\sinh^2\delta_2)}{(r + 2m\sinh^2\delta_3)(r + 2m\sinh^2\delta_4)} \right]^{\frac{1}{2}} \ , \\
&A_I = \frac{2m \sinh\delta_I \cosh\delta_I}{r + 2m\sinh^2\delta_I} \, dt, \quad (\mathrm{I} = 1, 2, 3, 4) \, .
\end{aligned} \tag{10}$$

2.1 Static Case

We shall first demonstrate the scaling limit, leading to the subtracted geometry of the general *static* solution. We perform a scaling limit on the static solutions (7)-(10) where without loss of generality we take three equal charges and the fourth one different by defining $*F_1 = F_2 = *\mathcal{F}_1 \equiv F$ and $\mathcal{F}_2 \equiv \mathcal{F}$. [3] We use the "tilde" notation for all the variables, with the choice of charge parameters $\tilde\delta_1 = \tilde\delta_2 = \tilde\delta_3 \equiv \tilde\delta$ and $\tilde\delta_4 \equiv \tilde\delta_0$. We take the following scaling limit with $\epsilon \to 0$:

$$\begin{aligned}
&\tilde r = r\epsilon, \quad \tilde t = t\epsilon^{-1}, \quad \tilde m = m\epsilon \, , \\
&2\tilde m \sinh^2\tilde\delta \equiv Q = 2m\epsilon^{-1/3}(\Pi_c^2 - \Pi_s^2)^{1/3}, \quad \sinh^2\tilde\delta_0 = \frac{\Pi_s^2}{\Pi_c^2 - \Pi_s^2} \, ,
\end{aligned} \tag{11}$$

where the "tilde" coordinates and parameters of the scaled solution are related to those of the subtracted geometry for the four-charge static black hole. In the latter case the metric of the (unsbtracted) black hole solution is of the form (7), but with the subtracted geometry the metric (7) is the same, except for the warp factor:

$$\Delta_{0s} \to \Delta_s = (2m)^3 r (\Pi_c^2 - \Pi_s^2) + (2m)^4 \Pi_s^2 \ . \tag{12}$$

The sources supporting this geometry are obtained by taking the scaling limit (11) in (10) (with "tilde" coordinates and parameters):

$$\chi_1 = \chi_2 = \chi_3 = 0, \quad e^{\varphi_1} = e^{\varphi_2} = e^{\varphi_3} = \frac{Q^2}{\Delta_s^{\frac{1}{2}}},$$

[3] While one can in principle perform a scaling limit with three unequal large charges Q_i ($I = 1, 2, 3$), by replacing in the scaling limit (11) $Q \to (\Pi_{I=1}^3 Q_I)^{\frac{1}{3}}$, appropriate powers of Q_I in the scalar fields φ_i ($i = 1, 2, 3$) and gauge field strengths $*F_1, F_2, *\mathcal{F}_1$ can be removed without loss of generality, resulting in the same gauge choice for sources (15).

$$A=-\frac{r}{Q}\,dt,\quad \mathcal{A}=\frac{Q^3(2m)\Pi_c\Pi_s}{(\Pi_c^2-\Pi_s^2)\Delta_s}\,dt, \tag{13}$$

resulting in field strengths:

$$F_{t\,r}=\frac{1}{Q},\quad \mathcal{F}_{t\,r}=\frac{Q^3(2m)^4\Pi_c\Pi_s}{\Delta_s^2}\,. \tag{14}$$

The (formally infinite) factors of Q can be removed with sources taking the form:

$$\begin{aligned}&\chi_1=\chi_2=\chi_3=0,\quad e^{\varphi_1}=e^{\varphi_2}=e^{\varphi_3}=\frac{(2m)^2}{\Delta_s^{\frac{1}{2}}}\,,\\ &A=-\frac{r}{2m}\,dt,\quad \mathcal{A}=\frac{(2m)^4\Pi_c\Pi_s}{(\Pi_c^2-\Pi_s^2)\Delta_s}\,dt,\end{aligned} \tag{15}$$

with electric field strengths:

$$F_{t\,r}=\frac{1}{2m}\,,\quad \mathcal{F}_{t\,r}=\frac{(2m)^7\Pi_c\Pi_s}{\Delta_s^2}\,. \tag{16}$$

The result for sources is the same (up to a gauge choice) as the one obtained in [13] by directly solving Einstein equations with the subtracted geometry static metric. Note that the sources supporting this geometry are those of the minimal supergravity in five dimensions, where F is the Maxwell field strength of the five-dimensional theory and $\mathcal{F}$ the Kaluza-Klein field strength.

2.2 General Rotating Case

We now proceed with obtaining a subtracted geometry for a general four-charge *rotating* black hole, whose original metric (3) was displayed at the beginning of this section.

In order to track the effects associated with the rotational parameter a in the scaling limit we display explicitly the metric and the sources for the solution (transcribed from [15]) with three equal charges and the fourth one different, i.e., by again choosing, without loss of generality, the gauge potentials $A_1=A_2=A_3\equiv A$ for gauge field strengths $*F_1=F_2=*\mathcal{F}_1\equiv F$ and $A_4\equiv\mathcal{A}$ for $\mathcal{F}_2\equiv\mathcal{F}$. The metric is written as above (3), but with all the quantities taken with "tilde" notation and denoting $\tilde\delta_1=\tilde\delta_2=\tilde\delta_3\equiv\tilde\delta$ and $\tilde\delta_4\equiv\tilde\delta_0$.

The scalar fields are given by:

$$\begin{aligned}\chi_1=\chi_2=\chi_3&=\frac{2\tilde m\,\tilde a\cos\theta\,\cosh\tilde\delta\sinh\tilde\delta(\cosh\tilde\delta\sinh\tilde\delta_0-\sinh\tilde\delta\cosh\tilde\delta_0)}{(\tilde r+2\tilde m\sinh^2\tilde\delta)^2+\tilde a^2\cos^2\theta},\\ e^{\varphi_1}=e^{\varphi_2}=e^{\varphi_3}&=\frac{(\tilde r+2\tilde m\sinh^2\tilde\delta)^2+\tilde a^2\cos^2\theta}{\tilde\Delta_0^{\frac{1}{2}}}\,,\end{aligned} \tag{17}$$

and the gauge potentials by:

$$\begin{aligned}A=\;&\frac{2\tilde m}{\tilde\Delta_0}\{[(\tilde r+2\tilde m\sinh^2\tilde\delta)^2(\tilde r+2\tilde m\sinh^2\tilde\delta_0)+\tilde r\tilde a^2\cos^2\theta][\cosh\tilde\delta\sinh\tilde\delta\,d\tilde t\\ &-\tilde a\sin^2\theta\cosh\tilde\delta\sinh\tilde\delta(\cosh\tilde\delta\cosh\tilde\delta_0-\sinh\tilde\delta\sinh\tilde\delta_0)\,d\phi]\\ &+2\tilde m\,\tilde a^2\cos^2\theta\,[e\,d\tilde t-\tilde a\sin^2\theta\sinh^2\tilde\delta\cosh\tilde\delta\sinh\tilde\delta_0\,d\phi]\}\,,\\ \mathcal{A}=\;&\frac{2\tilde m}{\tilde\Delta_0}\{[(\tilde r+2\tilde m\sinh^2\tilde\delta)^3+\tilde r\tilde a^2\cos^2\theta][\cosh\tilde\delta_0\sinh\tilde\delta_0\,d\tilde t\\ &-\tilde a\sin^2\theta(\cosh^3\tilde\delta\sinh\tilde\delta_0-\sinh^3\tilde\delta\cosh\tilde\delta_0)\,d\phi]\end{aligned}$$

$$+2\tilde{m}\,\tilde{a}^2\cos^2\theta\,[e_0\,d\tilde{t} - \tilde{a}\sin^2\theta\sinh^3\tilde{\delta}\cosh\tilde{\delta}_0\,d\phi]\}\,. \tag{18}$$

Here:

$$\begin{aligned} e &= \sinh^2\tilde{\delta}\cosh^2\tilde{\delta}\cosh\tilde{\delta}_0\sinh\tilde{\delta}_0(\cosh^2\tilde{\delta}+\sinh^2\tilde{\delta}) \\ &\quad -\sinh^3\tilde{\delta}\cosh\tilde{\delta}(\sinh^2\tilde{\delta}+2\sinh^2\tilde{\delta}_0+2\sinh^2\tilde{\delta}\sinh^2\tilde{\delta}_0), \\ e_0 &= \sinh^3\tilde{\delta}\cosh^3\tilde{\delta}(\cosh^2\tilde{\delta}_0+\sinh^2\tilde{\delta}_0)-\sinh\tilde{\delta}_0\cosh\tilde{\delta}_0(3\sinh^4\tilde{\delta}+2\sinh^6\tilde{\delta})\,. \end{aligned} \tag{19}$$

Again, we take the scaling limit (11), and furthermore we take for the rotational parameter:

$$\tilde{a} = a\epsilon\,. \tag{20}$$

In terms of new coordinates and parameters the metric takes the form (3), where only the warp factor changes:

$$\Delta_0 \to \Delta = (2m)^3 r(\Pi_c^2-\Pi_s^2) + (2m)^4\Pi_s^2 - (2m)^2 a^2(\Pi_c-\Pi_s)^2\cos^2\theta\ . \tag{21}$$

This geometry with the subtracted warp factor is sourced by the scalars:

$$\chi_1=\chi_2=\chi_3=-\frac{2ma(\Pi_c-\Pi_s)cos\theta}{Q^2},\quad e^{\varphi_1}=e^{\varphi_2}=e^{\varphi_3}=\frac{Q^2}{\Delta^{\frac{1}{2}}}\,, \tag{22}$$

and the gauge potentials:

$$\begin{aligned} A = &\quad -\frac{r}{Q}dt + \frac{(2m)^2a^2[2m\Pi_s^2 - r(\Pi_c-\Pi_s)^2]\cos^2\theta}{Q\Delta}dt \\ &\quad -\frac{2m\,a(\Pi_c-\Pi_s)\sin^2\theta}{Q}\left(1+\frac{(2m)^2a^2(\Pi_c-\Pi_s)^2\cos^2\theta}{\Delta}\right)d\phi\,, \\ \mathcal{A} = &\quad \frac{Q^3[(2m)^2\Pi_c\Pi_s + a^2(\Pi_c-\Pi_s)^2\cos^2\theta]}{2m(\Pi_c^2-\Pi_s^2)\Delta}\,dt\,+\,\frac{Q^3 2m\,a(\Pi_c-\Pi_s)\sin^2\theta}{\Delta}\,d\phi\,, \end{aligned} \tag{23}$$

resulting in field strengths with both electric and magnetic components. The (formally infinite) factors of Q can again be removed from gauge potentials by removing corresponding factors from scalar fields, and thus the sources take the canonical form:

$$\chi_1=\chi_2=\chi_3=-\frac{a(\Pi_c-\Pi_s)\cos\theta}{2m},\quad e^{\varphi_1}=e^{\varphi_2}=e^{\varphi_3}=\frac{(2m)^2}{\Delta^{\frac{1}{2}}}\,, \tag{24}$$

$$\begin{aligned} A = &\quad -\frac{r}{2m}dt + \frac{(2m)a^2[2m\Pi_s^2 - r(\Pi_c-\Pi_s)^2]\cos^2\theta}{\Delta}dt \\ &\quad -a(\Pi_c-\Pi_s)\sin^2\theta(1+\frac{(2m)^2a^2(\Pi_c-\Pi_s)^2\cos^2\theta}{\Delta})\,d\phi, \\ \mathcal{A} = &\quad \frac{(2m)^4\Pi_c\Pi_s + (2m)^2a^2(\Pi_c-\Pi_s)^2\cos^2\theta}{(\Pi_c^2-\Pi_s^2)\Delta}\,dt\,+\,\frac{(2m)^4a(\Pi_c-\Pi_s)\sin^2\theta}{\Delta}\,d\phi\,, \end{aligned} \tag{25}$$

Again, the sources supporting this geometry are those of the minimal supergravity in five-dimensions, where F and $\mathcal{F}$ are the five-dimensional Maxwell and Kaluza-Klein field strengths, respectively.

The scaling limits (11),(20) are reminiscent of the near-BPS dilute gas approximation [8], which were generalized to rotating four-dimensional black holes in [18]. As a natural consequence, the subtracted geometry of general black holes is a Kaluza-Klein coset of $AdS_3 \times 4S^2$ just as in the dilute-gas approximation [18]. Furthermore, there is an analogous microscopic interpretation

in terms of two-dimensional conformal field theory of a long rotating string, which was addressed in [13].

In Appendix A we show that the subtracted geometry of general five-dimensional black holes can be obtained as an analogous scaling limit, reminiscent of the near-BPS dilute gas approximate for five-dimensional rotating black holes [19], resulting in the Kaluza-Klein coset of $AdS_3 \times S^3$, and analogous microscopic interpretation via a conformal field theory of a long rotating string, studied in [12]. In Section 4 we further analyse geometric properties of emerging Kaluza-Klein cosets.

2.3 Subtracted Geometry as a Harrison Transformation

In this Subsection we demonstrate that the subtracted geometry can be obtained as a ic Harrison transformation on the original black hole solution. For the sake of simplicity and in order to demonstrate the procedure we shall present the details for the Schwarzschild black hole, only. In this case it is sufficient to employ the Einstein-Dilaton-Maxwell Lagrangian density, with the dilation coupling $\alpha = \frac{1}{\sqrt{3}}$, which is a consistent truncation of the Lagrangian density (1) with $\chi_i = 0$, $\varphi_i = \varphi_2 = \varphi_3 \equiv -\frac{2}{\sqrt{3}}\phi$, $*F_1 = F_2 = *\mathcal{F}_1 \equiv \sqrt{\frac{2}{3}}F$ and $\mathcal{F}_2 = 0$. [Of course for the multi-charged rotating black holes one has to employ the full N=2 supergravity Lagrangian density (1).]

We begin by considering static solutions to general Einstein-Dilaton-Maxwell equations with the general dilation coupling α. The Lagrangian density is [4].:

$$\sqrt{-g}\Big(\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2\alpha\phi}F^2\Big)\,. \tag{26}$$

Making the Ansatz

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\gamma_{ij}dx^i dx^j\,, \qquad F_{i0} = \partial_i\psi \tag{27}$$

we obtain an effective action density in three dimensions of the form

$$\sqrt{\gamma}\Big(R(\gamma_{ij}) - 2\gamma^{ij}\Big(\partial_i U\partial_j U + \partial_i\phi\partial_j\phi - e^{-2U}e^{-2\alpha\phi}\partial_i\psi\partial_j\psi\Big)\Big) \tag{28}$$

Defining

$$x \equiv \frac{U+\alpha\phi}{\sqrt{1+\alpha^2}}\,, \qquad y \equiv \frac{-\alpha U+\phi}{\sqrt{1+\alpha^2}}\,, \tag{29}$$

the effective action density becomes

$$\sqrt{\gamma}\Big(R(\gamma_{ij}) - 2\gamma^{ij}\Big(\partial_i x\partial_j x + \partial_i y\partial_j y - e^{-2\sqrt{1+\alpha^2}x}\partial_i\psi\partial_j\psi\Big)\Big)\,. \tag{30}$$

Evidently we can consistently set $y = 0$ and we obtain a sigma model, whose fields x, ψ map into the target $SL(2,\mathbb{R})/SO(1,1)$, coupled to three dimensional Einstein gravity. The non-trivial action of an $SO(1,1)$ subgroup of $SL(2,\mathbb{R})$ is called a Harrison transformation.

More concretely, and following [20] but making some changes necessitated by considering a reduction on time-like, rather than a space-like Killing vector we define a matrix (See also, e.g., [21] and references therein.):

$$P \;=\; e^{-\sqrt{1+\alpha^2}(x+y)}\begin{pmatrix} e^{2\sqrt{1+\alpha^2}x} - (1+\alpha^2)\psi^2 & -\sqrt{1+\alpha^2}\psi \\ -\sqrt{1+\alpha^2}\psi & -1 \end{pmatrix}\,, \tag{31}$$

[4]We choose the units in which $4\pi G = 1$. Note that in the Lagrangian density (1) $16\pi G = 1$ and the field strengths differ by a factor of $\sqrt{2}$.

so that

$$P = P^T\,, \quad \det P = -e^{-2\sqrt{1+\alpha^2}y}\,. \tag{32}$$

Taking $H \in SO(1,1)$ which acts on P as

$$P \to HPH^T\,, \tag{33}$$

it preserves not only the properties (32) but also the Lagrangian density (30) which can be cast in the form:

$$\sqrt{\gamma}\Big(R(\gamma_{ij}) + \frac{1}{1+\alpha^2}\gamma^{ij}\mathrm{Tr}(\partial_i P \partial_j P^{-1})\Big). \tag{34}$$

It is straightforward to show that a Harrison transformation:

$$H = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}\,, \tag{35}$$

corresponds to:

$$\begin{aligned} y' &= y\,, \qquad e^{\sqrt{1+\alpha^2}x'} = \Lambda^{-1} e^{\sqrt{1+\alpha^2}x}\,, \\ \psi' &= \Lambda^{-1}[\psi + \frac{\beta}{\sqrt{1+\alpha^2}}(e^{2\sqrt{1+\alpha^2}x} - (1+\alpha^2)\psi^2)]\,; \ \Lambda = (\beta\psi+1)^2 - \beta^2 e^{2\sqrt{1+\alpha^2}x}\,. \end{aligned} \tag{36}$$

Note, this transformation can also be determined as an analytic continuation of transformations given in Section 2 of [20]. A Harrison transformation in the limit of an infinite boost corresponds to $\beta \to 1$. One may verify that (35) with $\beta \to 1$ in the Einstein-Maxwell gravity ($\alpha = 0$) takes the Schwarzschild metric to the Robinson-Bertotti one. This type of transformation was employed recently in [22]. For another work, relating the Schwarzschild geometry to $AdS_2 \times S^2$, see [23].

In the case of $\alpha = \frac{1}{\sqrt{3}}$, we shall act with (35) on the Schwarzschild solution with $e^{2U} = 1 - \frac{2m}{r}$, $\phi = 0$, $\psi = 0$. The transformation (36) with $\beta = 1$ results in $\Lambda = \frac{2m}{r}$, and the metric (7) with the subtracted geometry warp factor:

$$\Delta_{s0} = r^4 \to \Delta_s = (2m)^3 r\,, \tag{37}$$

and the scalar field and the electric field strength :

$$e^{-\frac{2\phi}{\sqrt{3}}} = \sqrt{\frac{2m}{r}}\,, \qquad \sqrt{\frac{2}{3}}F_{t\,r} = \frac{1}{2m}\,, \tag{38}$$

i.e., this is the static subtracted geometry of Subsection 2.1, with $\Pi_c = 1,\ \Pi_s = 0$.

The subtracted geometry for the Kerr spacetime can be obtained by reducing the spacetime on the time-like Killing vector and acting on the Kerr black hole with an infinite boost Harrison transformation for Lagrangian density (1), where we set $\chi_1 = \chi_2 = \chi_3 \equiv \chi$, $\varphi_1 = \varphi_2 = \varphi_3 \equiv \frac{2}{\sqrt{3}}\phi$, $*F_1 = F_2 = *\mathcal{F}_1 \equiv \sqrt{\frac{2}{3}}F$ and $\mathcal{F}_2 = \sqrt{2}\mathcal{F}$, i.e. an Einstein-Dilaton-Axion gravity with two $U(1)$ gauge fields and respective dilaton couplings $\alpha_1 = \frac{1}{\sqrt{3}}$ and $\alpha_2 = \sqrt{3}$. The subtracted geometry of the multi-charged rotating black holes is expected to arise as a ic Harrison transformation on a rotating charged black solution of (1). This has recently been confirmed [14].

These results demonstrate that the subtracted geometry is a solution of the same theory as the original black hole. Furthermore the original black hole and the subtracted geometry clearly lie in the same duality orbit, specified above and passing through the original black hole. Thus any physical property of the original black hole solution which is invariant under the duality transformation of M-theory remains the property of the subtracted geometry. For example the area of the horizon is unchanged.

3 Asymptotically Conical Metrics

The scaling limit, or equivalently the subtraction process, alters the environment that our black holes find themselves in [12, 13]. In fact the subtracted geometry metric is asymptotically of the form

$$ds^2 = -\big(\frac{R}{R_0}\big)^{2p} dt^2 + B^2 dR^2 + R^2\big(d\theta^2 + \sin^2\theta^2 d\phi^2)\big) \tag{39}$$

with R_0 a constant, $B = 4$ and $p = 3$. In general, metrics with asymptotic form (39) may be referred as *Asymptotically Conical* (AC). The spatial metric is conical because the radial distance BR is a non-trivial multiple of the area distance R. Restricted to the equatorial plane the spatial metric is that of a flat two-dimensional cone

$$ds^2_{\rm equ} = B^2 dR^2 + R^2 d\phi^2 \tag{40}$$

with deficit angle

$$2\pi(1-\frac{1}{B}) = 8\pi\eta^2\,. \tag{41}$$

A characteristic feature of conical metrics of the form (39) is that they admit a *Lifshitz scaling.* That is they admit a *homothety*, a diffeomorphism under which the pulled back metric goes into a constant multiple of itself. In our case the homethety is

$$R \to \lambda R\,, \qquad t \to \lambda^{1-p} t\,, \qquad \Rightarrow \qquad ds^2 \to \lambda^2 ds^2 \tag{42}$$

or if one introduces isotropic Cartesian coordinates $\mathbf{x} = R^B(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ which render the spatial metric conformally flat, we have scaling under,

$$\mathbf{x} \to \lambda\mathbf{x}\,, \qquad t \to \lambda^z t\,, \tag{43}$$

where the difference of the Lifshitz scaling exponent z from unity is a measure of how space and time scale differently. We have

$$z = \frac{1-p}{B} \tag{44}$$

and so $z = -\frac{1}{2}$ in our case. The nonstandard scaling of time is reminiscent of the Lifshitz symmetry that has recently been developed for applications of holography to condensed matter systems. (See, e.g., [24, 25, 26] and references therein.).

Since asymptotically Conical (AC) metrics of the type (39) may be unfamiliar to the string theory community, we briefly recall some of their properties and give some examples. AC metrics typically arise when the energy density T_{00} of a static four dimensional spacetime falls off as $T_{00} \to \frac{\eta^2}{R^2}$ [5]. Because of this slow fall off the metric cannot have finite total energy and cannot be *asymptotically flat* (AF). At large distances such spacetime metrics typically take the form of (39) near infinity.

Examples where (39) is exact are

- $p = \frac{2\gamma}{1+\gamma}$, $B = \frac{\sqrt{1+6\gamma+\gamma^2}}{1+\gamma}$, gives *Bisnovatyi-Kogan Zeldovich's gas sphere* [27, 28] Here, γ is the constant ratio of pressure to density of. the gas for which

$$P \propto \frac{\gamma^2}{1+6\gamma+\gamma^2}\frac{1}{2\pi R^2} \tag{45}$$

[5]These components in a pseudo-orthonormal frame. The coordinate R is the area distance. We use throughout signature $-+++$ and units in which Newton's constant $G = 1$.

- $p=0$ and $B=\sqrt{1-8\pi\eta^2}$, gives the Barriola-Vilenkin Global Monopole [29]. The source if this metric is an $SO(3)$ non-linear sigma model with the Higgs field of constant magnitude η in the hedgehog configuration.
- The near horizon geometry of an extreme black hole in Einstein-Dilaton-Maxwell gravity with coupling constant α has $p=\frac{1}{\alpha^2}$, $z=1-\frac{1}{\alpha^2}$ and $B^2=\frac{1+\alpha^2}{\alpha^2}$. Our case corresponds to $\alpha^2=\frac{1}{3}$.

Examples of metrics which are asymptotically conical rather than being exactly conical are that of a black hole containing a global monopole [29] possibly with a magnetic (or electric) charge [30] for which

$$ds^2=-\big(1-8\pi\eta^2-\frac{2m}{R}+\frac{P^2}{R^2}\big)dt^2+\frac{dR^2}{1-8\pi\eta^2-\frac{2m}{R}+\frac{P^2}{R^2}}+R^2\big(d\theta^2+\sin^2\theta d\phi^2\big)\,. \tag{46}$$

One may also combine a global monopole with a Kaluza-Klein monopole [31] with four-dimensional metric

$$ds^2=-\big(1-8\pi\eta^2-\frac{2m}{R}\big)^{\frac{1}{2}}dt^2+\frac{dR^2}{\big(1-8\pi\eta^2-\frac{2m}{R}\big)^{\frac{1}{2}}}+\big(1-8\pi\eta^2-\frac{2m}{R}\big)^{\frac{1}{2}}R^2\big(d\theta^2+\sin^2\theta d\phi^2\big)\,. \tag{47}$$

A rather difference example is obtained by dimensional reduction of an 8-dimensional ultra static metric whose spatial metric is an asymptotically a cone over $S^3\times S^3$ with holonomy G_2 [32]. This has $B^2=\frac{4}{3}$ and $p=\frac{2}{3}$.

3.1 Flat Spacetime Analogs

In addition to having energy-densities falling off as R^{-2}, the subtracted geometries have the property that scalar fields do not tend to a constant at spatial infinity (24), but they run logarithmically at large R. Despite of this fact the total charges are finite. In fact this somewhat unfamiliar behavior can occur in general Einstein-Dilaton-Maxwell theory. Moreover, it is possible to take a limit in which gravity decouples and one discovers very similar behaviour in Dilaton-Maxwell theory [33]. Although the relevant solution has infinite energy in flat spacetime, the Maxwell and dilaton fields are perfectly regular outside the origin and they possess finite total electric or magnetic charge. Since their properties resemble many of their fully self-gravitating cousins we provide below a brief self-contained derivation. Of particular interest is the fact that the dilaton ϕ in these theories provides a spacetime dependent abelian gauge coupling constant g [6]:

$$g=e^{\alpha\phi}\,, \tag{48}$$

which in the static solutions "runs" from zero to infinity (magnetic case) or infinity to zero (electric case) as the radius r runs from zero to infinity. A similar running is seen in the fully self-gravitating solutions. The details are as follows.

The flat spacetime Lagrangian density:

$$4\pi\mathcal{L}=-\frac{1}{2}(\partial\phi)^2-\frac{1}{4}e^{-2\alpha\phi}F^2_{\mu\nu}\,, \tag{49}$$

leads to the following equation for the static dilaton field:

$$\boldsymbol{\nabla}^2\phi-\alpha e^{-2\alpha\phi}\mathbf{E}^2+\alpha e^{2\alpha\phi}\mathbf{B}^2=0\,, \tag{50}$$

[6] Equivalently one has spacetime dependent magnetic permeability $\mu=g^2$ and electric permittivity $\epsilon=g^{-2}$.

and the electric and magnetic field Ansätze:

$$D_r = E_r e^{-2\alpha\phi} = \frac{Q}{r^2}\,, \qquad B_r = \frac{P}{r^2}\,. \tag{51}$$

Thus

$$\boldsymbol{\nabla}^{\,2}\phi = \frac{1}{r^4}\alpha\big(Q^2 e^{2\alpha\phi} - P^2 e^{-2a\phi}\big)\,. \tag{52}$$

Defining $t \equiv \ln r$, we find:

$$\ddot{\phi} = \alpha\big(Q^2 e^{2\alpha\phi} - P^2 e^{-2\alpha\phi}\big)\,. \tag{53}$$

If $P = 0$ we try

$$\phi = A\ln r + B\,, \tag{54}$$

and find

$$A = \frac{1}{\alpha} = \alpha Q^2 e^{2\alpha B}\,. \tag{55}$$

If $Q = 0$ we try

$$\phi = A\ln r + B\,. \tag{56}$$

and find

$$A = -\frac{1}{\alpha} = -\alpha P^2 e^{-2\alpha B}\,. \tag{57}$$

In both cases

$$4\pi T_{00} = \frac{1}{\alpha^2 r^2}\,. \tag{58}$$

3.2 Confining Properties

It is helpful when discussing black hole thermodynamics to consider black hole confined in a box. One widely used example of such a box is provided by asymptotically anti-de Sitter spacetime. In this case the spatial geometry is asymptotically hyperbolic and the blue-shift factor $\sqrt{-g_{00}}$ increases exponentially with proper radial distance. This leads to confinement of massive particles and of thermal radiation. By Tolman's red-shifting formula for thermal radiation, the temperature $T \propto 1/\sqrt{-g_{00}}$ and thus falls off exponentially with proper distance. As a consequence the total energy and entropy outside any given radius are finite.

In the following we show that subtracted geometry metrics have similar properties, thus justifying our claim that these metrics represent black holes in confining boxes. Actually, this confining property is a feature of AC metrics in general (39) with $p > 1$.

Since $\sqrt{-g_{00}} \propto R^p$, the situation is qualitatively similar to the anti-de Sitter case. More quantitatively, it is instructive to consider the motion of light-rays in this background. Their spatial projections are geodesics of the optical metric

$$ds_o^2 = B^2\big(\frac{R_0}{R}\big)^{2p} dR^2 + R^2\big(\frac{R_0}{R}\big)^{2p}\big(d\theta^2 + \sin^2\theta d\phi^2\big) \tag{59}$$

If $\tilde{R} = R^{1-p}R_0^p$

$$ds_o^2 = (B')^2 d\tilde{R}^2 + \tilde{R}^2\big(d\theta^2 + \sin^2\theta d\phi^2\big) \tag{60}$$

with $B' = \frac{B}{|1-p|}$. The optical manifold is also a *cone over a 2-sphere.* However, whereas if $p < 1$, $\tilde{R}$ increases as R increases and so spatial infinity is at an infinite optical distance, if $p > 1$, then

$\tilde{R}$ decreases as R-increases and infinity is a cone $\tilde{R}=0$ at a finite optical distance. Restricted to the equatorial plane the optical metric is that of a flat two-dimensional cone

$$2\pi(1-\frac{1}{B'})=2\pi(1-\frac{|1-p|}{B})\,. \tag{61}$$

In our case, $p=3$ and so outwardly directed light rays with R will, unless strictly radial, spiral around the optical cone whose vertex is at infinity for a finite time and return inwardly directed. The strictly radial light rays will reach infinity in finite time. In this sense the environment acts rather like a box surrounding the horizon in a fashion reminiscent of black holes in asymptotically AdS spacetimes.

As far as the Tolman red shifting is concerned the temperature of thermal radiation falls of as R^{-p}, thus the total entropy outside R will be finite if $p>1$ which is clearly satisfied in our case with $p=3$.

4 Symmetries of the Subtracted Geometry

4.1 Breaking of $(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))/\mathbb{Z}_2\times SO(3)$ by Kaluza-Klein Reduction

All of the 4-dimensional subtracted geometries considered in this paper may be obtained [13] by reduction from the five-dimensional metric:

$$ds_5^2=ds^2_{AdS_3}+4ds^2_{S^2} \tag{62}$$

which is, up to a factor, the sum of the maximally symmetric metric on $AdS_3\equiv SL(2,\mathbb{R})$ of unit radius and the round metric on a unit 2-sphere. The isometry group of ds_5^2 is thus $SO(3)\times((SL(2,\mathbb{R})\times SL(2,\mathbb{R}))/\mathbb{Z}_2)$. In the general case the space-like Killing vector $\frac{\partial}{\partial\alpha}$ whose orbits the Kaluza-Klein reduction is effected generates a one parameter subgroup H of $G=SO(3)\times((SL(2\mathbb{R})\times SL(2,\mathbb{R}))/\mathbb{Z}_2)$ with projections in all three factors. Thus in general the quotient G/H admits an effective action of the centraliser of H in G which in general consists of just two Killing vectors ∂_t and ∂_ϕ. Nevertheless when solving for the massless wave equation in these geometries one discovers that the solutions may be expressed in terms of hypergeometric functions (see, e.g., [8, 5]) and indeed the wave operator may be expressed as a sum of the Casimir for $SL(2,\mathbb{R})$ and $SO(3)$. Moreover, [13] these Casmirs may be seen to commute with all of the generators of $SO(3)\times((SL(2\mathbb{R})\times SL(2,\mathbb{R}))/\mathbb{Z}_2)$. Since in general only two of these generators correspond to Killing fields, this is on the face of it puzzling. It suggest that perhaps the solutions of the massless wave equation on the subtracted geometries carry a representation of $SO(3)\times((SL(2\mathbb{R})\times SL(2,\mathbb{R}))/\mathbb{Z}_2)$. However this is manifestly not true. The reason being that while solutions of the massless wave equation on (62) do indeed carry a representation of $SO(3)\times((SL(2\mathbb{R})\times SL(2,\mathbb{R}))/\mathbb{Z}_2)$, *only those solutions invariant under the action of H descend to the quotient* $(AdS_3\times S^2)/H$. Thus there is no action of the full $G=SO(3)\times((SL(2\mathbb{R})\times SL(2,\mathbb{R}))/\mathbb{Z}_2)$ on solutions of the wave equation on the four-dimensional spacetime $(AdS_3\times S^2)/H$, but just of the centraliser of H in G, which is generated by precisely the four-dimensional Killing fields ∂_t and ∂_ϕ. In the following Subsection we verify these results by employing explicit expressions for the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ and $SO(3)$ generators in terms of four-dimensional coordinates (t,r,ϕ) and the fifth coordinate α.

4.2 Action of $SL(2,\mathbb{R})$ Generators

The lift of the subtracted geometry for general four-dimensional black holes to five dimensions on coordinate α is locally described [13] i as a metric (62) which is a sum (up to rescaling) of the maximally symmetric metric on $AdS_3 \equiv SL(2,\mathbb{R})$ of and the round metric on a unit 2-sphere. The isometry group of ds_5^2 is thus $SO(3) \times ((SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\mathbb{Z}_2)$.

The AdS metric can be written in global coordinates as:

$$ds_3^2 = \ell_{AdS}^2(d\rho^2 - \cosh^2\rho d\tau^2 + \sinh^2\rho d\sigma^2) \ . \tag{63}$$

The $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ generators $\mathcal{R}_i$ $(i = 1,2,3)$ and $\mathcal{L}_i$ $(i = 1,2,3)$ acting on AdS_3 are linear differential operators which commute with each other and satisfy $SL(2,\mathbb{R})$ algebras[7] :

$$[\mathcal{R}_i, \mathcal{R}_j] = 2i\epsilon_{ijk}(-)^{\delta_{k3}}\mathcal{R}_k \ \ ; \ [\mathcal{L}_i, \mathcal{L}_j] = 2i\epsilon_{ijk}(-)^{\delta_{k3}}\mathcal{L}_k \ , \tag{64}$$

and take the form:

$$\begin{aligned} \mathcal{R}_\pm &= \mathcal{R}_1 \pm i\mathcal{R}_2 = e^{\pm i(\tau+\sigma)}\left(\mp i\partial_\rho + \coth\rho\partial_\tau + \tanh\rho\partial_\sigma\right) \ , \\ \mathcal{R}_3 &= \partial_\tau + \partial_\sigma \ , \end{aligned} \tag{65}$$

and $\mathcal{L}_i$ determined by taking $\tau \to -\tau$.

The S^2 metric is written as:

$$ds_2^2 = \ell_S^2(d\Phi^2 + \sin^2\Theta d\Phi^2) \ , \tag{66}$$

where $SO(3)$ generators $\mathcal{J}_i$ (j=1,2,3) on S^2 satisfy the algebra:

$$[\mathcal{J}_i, \mathcal{J}_j] = \epsilon_{ijk}\mathcal{J}_k \ , \tag{67}$$

and take the form:

$$\begin{aligned} \mathcal{J}_\pm &= \mathcal{J}_1 \pm i\mathcal{J}_2 = e^{\pm i\Phi}\left(\partial_\Theta + \cot\Theta\partial_\Phi\right) \ , \\ \mathcal{J}_3 &= \partial_\Phi \ . \end{aligned} \tag{68}$$

In terms of the original coordinates t, r, θ, ϕ and $\tilde{\alpha}$ the metric (62) takes the form [13]:

$$\begin{aligned} ds_5^2 &= \Delta(d\tilde{\alpha} + \mathcal{B})^2 + \Delta^{-1/2}ds_4^2 \\ &= -\frac{X}{\rho}dt^2 + \frac{dr^2}{X} + \rho(d\tilde{\alpha} + \frac{\mathcal{A}_{\rm red}}{2m(\Pi_c - \Pi_s)\rho}dt)^2 + d\theta^2 + \sin^2\theta d\tilde{\beta}^2 \ , \end{aligned} \tag{69}$$

where ds_4^2 is the subtracted geometry four-dimensional metric (3) with $\Delta_0 \to \Delta$, Δ defined in (21), the Kaluza-Klein gauge potential $\mathcal{B}$ as defined in [13] and it is related to $\mathcal{A}$ in (25) as:

$$\mathcal{B} = \frac{\mathcal{A}}{(2m)^3} - \frac{dt}{(2m)^3(\Pi_c^2 - \Pi_s^2)} \ , \tag{70}$$

and

$$\begin{aligned} X &= r^2 - 2mr + a^2 \ , \\ \mathcal{A}_{\rm red} &= 2m[(\Pi_c - \Pi_s)r + 2m\Pi_s] \ , \\ \rho &= \mathcal{A}_{\rm red}^2 - 4m^2(\Pi_c - \Pi_s)^2 X = 8m^3[r(\Pi_c^2 - \Pi_s^2) + 2m\Pi_s^2 - \frac{a^2}{2m}(\Pi_c - \Pi_s)^2] \ , \end{aligned} \tag{71}$$

[7] The operators $\mathcal{R}_i$ and $\mathcal{L}_i$ are, in the usual way multiples by $-i$ of vector fields, so as to have real eigenvalues.

and

$$\tilde{\beta} = \phi + 2ma(\Pi_c - \Pi_s)\tilde{\alpha} \ . \tag{72}$$

The first part of the metric is AdS_3 with global coordinates expressed as:

$$\begin{aligned}
\sinh^2\rho &= \frac{r - r_+}{r_+ - r_-} \ , \quad r_\pm = m \pm \sqrt{m^2 - a^2} \, , \\
\sigma + \tau &= 2i\, m\sqrt{m^2 - a^2}(\Pi_c - \Pi_s)\, \tilde{\alpha} \, , \\
\sigma - \tau &= -i\frac{t}{2m(\Pi_c - \Pi_s)} - 2i\, m^2(\Pi_c + \Pi_s)\tilde{\alpha} \, ,
\end{aligned} \tag{73}$$

and $\ell_{AdS} = 2$, the results that can also be inferred from [13]. The generators $\mathcal{R}_3$ and $\mathcal{L}_3$ of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ now become:

$$\begin{aligned}
\mathcal{R}_3 &= -i\frac{1}{m\sqrt{m^2 - a^2}(\Pi_c - \Pi_s)}\partial_{\tilde{\alpha}} + 4i\frac{m^2(\Pi_c + \Pi_s)}{\sqrt{m^2 - a^2}}\partial_t \, , \\
\mathcal{L}_3 &= 4im(\Pi_c - \Pi_s)\partial_t \, .
\end{aligned}$$

We identify the compactification coordinate α with $\tilde{\alpha}$ and expressing the azymuthal four-dimensional angle ϕ in terms of $\tilde{\alpha}$ and $\tilde{\beta}$ as:

$$\begin{aligned}
\alpha &= \tilde{\alpha} \, , \\
\phi &= \tilde{\beta} - 2ma(\Pi_c - \Pi_s)\tilde{\alpha} \, ,
\end{aligned} \tag{74}$$

resulting in the following identification of the differential operators:

$$\begin{aligned}
\partial_{\tilde{\alpha}} &= \partial_\alpha - 2ma(\Pi_c - \Pi_s)\partial_\phi \, , \\
\partial_{\tilde{\beta}} &= \partial_\phi \, .
\end{aligned} \tag{75}$$

Consequently, the Cartan generators take the form:

$$\begin{aligned}
2\pi\mathcal{R}_3 &= -i\frac{2\pi}{m\sqrt{m^2 - a^2}(\Pi_c - \Pi_s)}\partial_\alpha + 2i\beta_H\Omega\partial_\phi + i\beta_R\partial_t \, , \\
2\pi\mathcal{L}_3 &= i\beta_L\partial_t \, ,
\end{aligned} \tag{76}$$

where β_H is the inverse Hawking temperature and Ω the angular velocity of the original black hole with the relation:

$$\beta_H\Omega = 2\pi\frac{a}{\sqrt{m^2 - a^2}} \, . \tag{77}$$

The inverse Hawking temperature β_H, and that at the inner horizon β_- can be expressed in terms of $\beta_{L,R}$:

$$\beta_H = \frac{1}{2}(\beta_R + \beta_L) \, , \quad \beta_- = \frac{1}{2}(\beta_R - \beta_L) \, , \tag{78}$$

where

$$\beta_R = 8\pi\frac{m^2}{\sqrt{m^2 - a^2}}(\Pi_c + \Pi_s) \, , \quad \beta_L = 8\pi m(\Pi_c - \Pi_s) \, . \tag{79}$$

On the other hand, the S^2 coordinates (66) are identified as:

$$\begin{aligned}
\Theta &= \theta \\
\Phi &= \tilde{\beta} = \phi + 2ma(\Pi_c - \Pi_s)\alpha \, ,
\end{aligned} \tag{80}$$

and $\ell_S = 1$. $\mathcal{J}_3$ generator of $SO(3)$ is just

$$\mathcal{J}_3 = \partial_\phi \,, \tag{81}$$

while $\mathcal{J}_\pm$ have an explicit dependence on ϕ and α.

The Laplacian on (70) is a sum of AdS_3 and S^2 Laplacians which can be cast in the form of the quadratic Casimirs of $SL(2,\mathbb{R})$ and $SO(3)$ generators, respectively. ally:

$$\begin{aligned}
\ell^2\nabla^2_{AdS_3} &= \mathcal{R}_1^2 + \mathcal{R}_2^2 - \mathcal{R}_3^2 \\
&= \mathcal{L}_1^2 + \mathcal{L}_2^2 - \mathcal{L}_3^2 \\
&= \frac{1}{\sinh 2\rho}\partial_\rho \sinh 2\rho \partial_\rho - \frac{1}{\cosh^2\rho}\partial_\tau^2 + \frac{1}{\sinh^2\rho}\partial_\sigma^2 \,, \qquad (82)\\
\ell^2\nabla^2_{S^2} &= \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2 \\
&= \frac{1}{\sin\Theta}\partial_\Theta(\sin\Theta\partial_\Theta) \ + \frac{1}{\sin^2\Theta}\partial_\Phi^2 \,, \qquad (83)
\end{aligned}$$

The AdS_3 Laplacian can be expressed in terms of partial derivatives ∂_r, ∂_t and $\partial_{\tilde\alpha} = \partial_\alpha - 2ma(\Pi_c - \Pi_s)\partial_\phi$, while the coefficients in front of derivatives depend on r, only. Similarly, the Laplacian on S^2 is expressed in terms of partial derivatives ∂_θ and ∂_ϕ, while the coefficients depend on θ, only. Therefore for fields which are *independent* of α, i.e. Kaluza-Klein reduced fields, the sum of the two Laplacians is the same as the Laplacians for the four-dimensional subtracted geometry metric, the fact noticed already in [13]. Note however, that this is the property of Laplacians, only. Namely, $\mathcal{R}_\pm$, $\mathcal{L}_\pm$ and $\mathcal{J}_\pm$, the raising and lowering generators of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$, have coefficients in front of derivatives which explicitly depend on α. These generators, when acting on α -independent fields necessarily transform them into α-dependent ones, and thus one has to study representations of fields under the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ actions in the full five-dimensional space. In other words, $\mathcal{R}_\pm$, $\mathcal{L}_\pm$ and $\mathcal{J}_\pm$ are not Killing vector fields of the subtracted metric, and thus do not generate isometries of the subtracted metrics.

4.3 Truncations of Five-dimensional Metric

We conclude this section with a few remarks on truncations of the five-dimensional metric (70). A truncation of this metric to the subspace $d\tilde\alpha = 0$ results in the four-dimensional metric:

$$ds^2_{5\,|\tilde\alpha} = \frac{dt^2}{4m^2(\Pi_c - \Pi_s)^2} + \frac{dr^2}{X} + d\theta^2 + \sin^2\theta\, d\phi^2 \,, \tag{84}$$

which is a product of the two-dimensional Euclidean space ($\mathbb{E}^2$) and a two-shere (S^2) with the isometry $\mathbb{R}^2 \rtimes SO(2) \times SO(3)$. When setting $d\tilde\alpha = 0$, the generators $\mathcal{L}_i$ turn to those of $E(2) = \mathbb{R}^2 \rtimes SO(2)$, the isometries of the two-dimensional Euclidean space $\mathbb{E}^2$. Note, however, that the truncation to $d\tilde\alpha = 0$ slice is not a Kaluza-Klein reduction of the five-dimensional space-time[8].

Another slice with $d\tilde\beta = 0$ reduces the five-dimensional metric (70) to $AdS_3 \times S^1/Z_2$. The $SO(2)$ generator is ∂_θ, and $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ generators can be expressed in terms of r,t, ϕ

[8]In the special case with $\mathcal{A} = 0$, a reduction on $d\tilde\alpha = 0$ results in the four-dimensional subtracted geometry is conformal to $\mathbb{E}^2 \times S^2$ of (70). These examples have $a = 0$, $\Pi_s = 0$ (see Eq.(25) for the explicit form of $\mathcal{A}$), i.e., those are subtracted geometries of static black holes with at least one zero charge and the conformal factor is $\Delta^{\frac{1}{2}} = (2m^3 r)^{\frac{1}{2}}\Pi_c$.

coordinates which are related to the global AdS_3 coordinates as:

$$\begin{aligned}
\sinh^2\rho &= \frac{r-r_+}{r_+-r_-}\ , \quad r_\pm = m \pm \sqrt{m^2-a^2}\ ,\\
\sigma+\tau &= -\frac{2\pi i}{\beta_H\Omega_H}\phi\ ,\\
\sigma-\tau &= -\frac{4\pi i}{\beta_L}(t-\frac{\beta_R}{2\beta_H\Omega_H}\phi)\ ,
\end{aligned} \tag{85}$$

and $\ell_{AdS}=2$. The above expressions are obtained by substituting $\tilde{a}=-\phi/[2ma(\Pi_c-\Pi_s)]$, a consequence of $d\tilde{\beta}=0$, in (73) and using Eqs.(77),(79). Note that those are coordinates typically quoted to specify generators $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ in the four-dimensional theory (see, e.g., [5, 11, 13]). However, this ation of the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ generators is in our case a consequence of a truncation of the five-dimensional metric to the slice $d\tilde{\beta}=0$ which is not a Kaluza-Klein reduction to the four-dimensional subtracted geometry.

5 Conclusions

In this paper we have addressed in further details the origin and properties of subtracted geometries for general multi-charged rotating charged black holes. These geometries where originally obtained [12, 13], by removing certain terms in the warp factor of the original metric in such a way that the massless wave equation exhibits the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})\times SO(3)$ symmetry. We showed that these geometries arise as a scaling limit of multi-charge rotating black holes with three large charges, reminiscent of the near-extreme multi-charge rotating black holes in the dilute gas approximation. (An analogous scaling limit for five-dimensional black holes is given in the Appendix.) The procedure also allows for the complete determination of sources. The subtracted geometry depends of on four parameters: bare mass m, bare angular parameter a, and two boost dependent products $\Pi_i\cosh\delta_i$ and $\Pi_i\sinh\delta_i$. Note that the original general black hole solution is determined by six parameters: bare mass m, bare angular parameter a and four independent boost parameters δ_i.

We have also shown that at least in the case of the Schwarzschild black hole the subtracted geometry can be obtained by performing a , "infinite boost" Harrison transformation in the Einstein-Dilaton-Maxwell gravity on the original unsubtracted black hole. In the previous version of this paper it was conjectured that the subtracted geometry of general multi-charged rotating black holes arises as a Harrison transformation of the original multi-charged rotating black hole. This has since been confirmed [14]. These results are significant since they show that the original black hole and the subtracted geometry clearly lie in the same orbit, specified by a Harrison transformation. Thus any physical property of the original black hole solution, which is invariant under duality transformation of the theory, remains a property of the subtracted geometry.

In retrospect, since the scaling limit is closely related to the dilute gas approximation, this elucidates the geometry as near-BPS and its origin as a Kaluza-Klein type quotient of $AdS_3\times 4S^2$ with $SL(2,\mathbb{R})\times SL(2,\mathbb{R})\times SO(3)$ symmetry manifest.

The subtracted geometry is asymptotically conical (AC), and it is reminiscent of the global monopole and the isothermal gas sphere behavior. Since the subtraction removes the ambient asymptotically Minkowski spacetime in a way that extracts the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})\times SO(3)$ symmetry of the black hole, it is dubbed "a black hole in an AC confining box". Since the subtracted metric has the same horizon area and periodicity's of the angular and time coordinates in the near horizon regions [12, 13] it is expected to preserve the internal structure of the black hole. An important further direction is a detailed investigation of its thermodynamic properties.

Acknowledgments We thank Finn Larsen for discussions and collaboration on related topics. G.W.G thanks the UPenn Department of Physics & Astronomy for hospitality. MC is supported by the DOE Grant DOE-EY-76-02- 3071, the NSF RTG DMS Grant 0636606, the Fay R. and Eugene L. Langberg Endowed Chair and the Slovenian Research Agency (ARRS).

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6 Appendix: Scaling limit for Five-Dimensional Black Holes

In this section, identify the full subtracted geometry of the general rotating black holes in five-dimensional $U(1)^3$ ungauged $\mathcal{N}=2$ supergravity.

The bosonic sector of the relevant $\mathcal{N}=2$ five-dimensional theory can be derived from the Lagrangian density:

$$e^{-1}\,\mathcal{L} = R - \frac{1}{2}\delta\vec{\varphi}^2 - \frac{1}{4}\sum_{i=1}^{3} X_i^{-2}\,(F^i)^2 + \frac{1}{24}|\epsilon_{ijk}|\,\epsilon^{\mu\nu\rho\sigma\lambda}F^i_{\mu\nu}\,F^j_{\rho\sigma}\,A^k_\lambda\,, \tag{86}$$

where $\vec{\varphi} = (\varphi_1, \varphi_2)$, and

$$X_1 = e^{-\frac{1}{\sqrt{6}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}\,, \qquad X_2 = e^{-\frac{1}{\sqrt{6}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}\,, \qquad X_3 = e^{\frac{2}{\sqrt{6}}\varphi_1}\,. \tag{87}$$

We write the 5D metric of the general rotating black hole [9] as a fibration over a 4D base space [35][10]

$$\begin{aligned} ds_5^2 &= -\Delta_0^{-2/3}G(dt+\mathcal{A})^2 + \Delta_0^{1/3}ds_4^2\,,\\ ds_4^2 &= \frac{dx^2}{4X} + \frac{dy^2}{4Y} + \frac{U}{G}(d\chi - \frac{Z}{U}d\sigma)^2 + \frac{XY}{U}d\sigma^2\,, \end{aligned} \tag{88}$$

where

$$\begin{aligned} \Delta_0 &= (x+y)^3 H_1H_2H_3\,, \quad X = (x+a^2)(x+b^2) - 2mx\,, \quad Y = -(a^2-y)(b^2-y)\,,\\ G &= (x+y)(x+y-2m)\,, \quad U = yX - xY\,, \quad Z = ab(X+Y)\,,\\ \mathcal{A} &= \frac{2m\Pi_c}{x+y-2m}[(a^2+b^2-y)d\sigma - abd\chi] - \frac{2m\Pi_s}{x+y}(abd\sigma - yd\chi)\,, \end{aligned} \tag{89}$$

The scalars are given by

$$X_i = H_i^{-1}\,(H_1H_2H_3)^{1/3}\,, \tag{90}$$

and gauge potentials by

$$\begin{aligned} A^1 =& \frac{2m}{(x+y)H_1}\{\sinh\delta_1\cosh\delta_1\,dt + \sinh\delta_1\cosh\delta_2\cosh\delta_3[abd\chi + (y-a^2-b^2)d\sigma]\\ +& \cosh\delta_1\sinh\delta_2\sinh\delta_3(abd\sigma - yd\chi)\}\,, \end{aligned} \tag{91}$$

where $A^2$and A^3 determined by acting with cyclic permutations on δ_i parameters in A^1. Here:

$$H_i = 1 + \frac{2m\sinh^2\delta_i}{x+y}, \quad (i=1,2,3), \tag{92}$$

and we have defined:

$$\Pi_c \equiv \prod_{i=1}^{3} = \cosh\delta_i, \quad \Pi_s \equiv \prod_{i=1}^{3}\sinh\delta_i\,. \tag{93}$$

[9]This three-charge rotating black hole is a generating solution for the most general charged rotating black hole of maximally supersymmetric give-dimensional $\mathcal{N}=4$ and $\mathcal{N}=8$ supergravity theory, which is can be obtained as a toroidal compactification on an effective heterotic string theory and Type IIA superstring theory, respectively. The most general charged rotating black hole can be obtained by acting on the generating solution with a subset of respective $\{S,T\}$- and U- duality transformations.

[10]The base space coordinates (x,y,σ,χ) are related to the more familiar radial and angular coordinates coordinates (r,θ,ϕ,ψ) as $x=r^2$, $y = a^2\cos^2\theta + b^2\sin^2\theta$, $\sigma = \frac{1}{a^2-b^2}\,(a\phi - b\psi)$, $\chi = \frac{1}{a^2-b^2}\,(b\phi - a\psi)$.

Note that the solution is parameterized by the bare mass m, two rotating parameters a, b and three charge parameters δ_i $(i = 1, 2, 3)$.

The subtracted geometry for these backgrounds is obtained by taking the scaling limit of the solution where we shall denote all the variables with "tilde" and without loss of generality taking large two charge parameters equal: $\tilde{\delta}_1 = \tilde{\delta}_2 \equiv \tilde{\delta}$. The coordinates and the parameters scale with $\epsilon \to 0$ as:

$$\begin{aligned}
&\tilde{x} = x\epsilon, \quad \tilde{t} = t\epsilon^{-1}, \quad \tilde{y} = y\epsilon, \quad \tilde{\sigma} = \sigma\epsilon^{-1/2}, \quad \tilde{\chi} = \chi\epsilon^{-1/2}, \\
&\tilde{m} = m\epsilon, \quad \tilde{a}^2 = a^2\epsilon, \quad \tilde{b}^2 = b^2\epsilon, \\
&2\tilde{m}\sinh^2\tilde{\delta} \equiv Q = 2m\epsilon^{-1/2}(\Pi_c^2 - \Pi_s^2)^{1/2}, \quad \sinh^2\tilde{\delta}_3 = \frac{\Pi_s^2}{\Pi_c^2 - \Pi_s^2}
\end{aligned} \tag{94}$$

The subtracted geometry metric has the same form (88) as the general black hole solution except for the subtracted warp factor:

$$\Delta_0 \to \Delta = (2m)^2(x+y)(\Pi_c^2 - \Pi_s^2) + (2m)^3\Pi_s^2 \,. \tag{95}$$

This geometry is sourced by the scalar fields:

$$X_1 = X_2 = X_3^{-\frac{1}{2}} = \frac{\Delta^{\frac{1}{3}}}{2m}\,, \tag{96}$$

and the gauge potentials:

$$\begin{aligned}
A^1 &= A^2 = -\frac{x+y}{2m}\,dt + y\Pi_c\,d\sigma - y\Pi_s\,d\chi\,, \\
A^3 &= \frac{(2m)^4\Pi_s\Pi_c}{(\Pi_c^2 - \Pi_s^2)\Delta}\,dt + \frac{\Pi_s}{\Delta}[ab\,d\chi + (y - a^2 - b^2)d\sigma] + \frac{\Pi_c}{\Delta}(ab\,d\sigma - y\,d\chi)\,.
\end{aligned} \tag{97}$$

Note that we have chosen a gauge where we have rescaled the scalars and the field strengths by appropriate factors of ϵ and $\Pi_c^2 - \Pi_s^2$. The solution is of co-homogeneity two, with gauge field strengths having both electric and magnetic components.

The scaling limit, reminiscent of the dilute gas approximation, extracts the subtracted geometry of the five-dimensional black hole which is a Kaluza-Klein coset of $AdS_3 \times S^3$ exhibiting conformal invariance. It is a solution of the six-dimensional Lagrangian (2) with F^3 corresponding to the Kaluza-Klein field strength. The scaling limit also signifies that the geometry is supersymmetric.